Journal of Combinatorial Theory, Series B 100 (2010) 332–346

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Dirac-type results for loose Hamilton cycles in uniform hypergraphs

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ARTICLE INFO

Article history:
Received 24 July 2008
Available online 22 November 2009

Keywords:
Hypergraphs
Hamilton cycles
Dirac's theorem

ABSTRACT

A classic result of G.A. Dirac in graph theory asserts that for $n \geqslant 3$ every n-vertex graph with minimum degree at least $n/2$ contains a spanning (so-called *Hamilton*) cycle. G.Y. Katona and H.A. Kierstead suggested a possible extension of this result for k-uniform hypergraphs. There a Hamilton cycle of an n-vertex hypergraph corresponds to an ordering of the vertices such that every k consecutive (modulo n) vertices in the ordering form an edge. Moreover, the minimum degree is the minimum $(k-1)$-degree, i.e. the minimum number of edges containing a fixed set of $k-1$ vertices. V. Rödl, A. Ruciński, and E. Szemerédi verified (approximately) the conjecture of Katona and Kierstead and showed that every n-vertex, k-uniform hypergraph with minimum $(k-1)$-degree $(1/2+o(1))n$ contains such a *tight* Hamilton cycle. We study the similar question for Hamilton ℓ-cycles. A Hamilton ℓ-cycle in an n-vertex, k-uniform hypergraph $(1 \leqslant \ell < k)$ is an ordering of the vertices and an ordered subset of the edges such that each such edge corresponds to k consecutive (modulo n) vertices and two consecutive edges intersect in precisely ℓ vertices.

We prove sufficient minimum $(k-1)$-degree conditions for Hamilton ℓ-cycles if $\ell < k/2$. In particular, we show that for every $\ell < k/2$ every n-vertex, k-uniform hypergraph with minimum $(k-1)$-degree $(1/(2(k-\ell))+o(1))n$ contains such a *loose* Hamilton ℓ-cycle. This degree condition is approximately tight and was conjectured by D. Kühn and D. Osthus (for $\ell = 1$), who verified it when $k = 3$. Our proof is based on the so-called weak regularity

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[1] Supported by DFG within the research training group "Methods for Discrete Structures".

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2009.10.002

lemma for hypergraphs and follows the approach of V. Rödl, A. Ruciński, and E. Szemerédi.

1. Introduction

We consider k-uniform hypergraphs $\mathcal{H}$, that are pairs $\mathcal{H} = (V, E)$ with vertex sets $V = V(\mathcal{H})$ and edge sets $E = E(\mathcal{H}) \subseteq \binom{V}{k}$, where $\binom{V}{k}$ denotes the family of all k-element subsets of the set V. We often identify a hypergraph $\mathcal{H}$ with its edge set, i.e. $\mathcal{H} \subseteq \binom{V}{k}$. Given a k-uniform hypergraph $\mathcal{H} = (V, E)$ and a set $S \in \binom{V}{s}$ let $\deg(S)$ denote the number of edges of $\mathcal{H}$ containing the set S and let $\delta_s(\mathcal{H})$ be the minimum s-degree of $\mathcal{H}$, i.e. the minimum of $\deg(S)$ over all s-element sets $S \subseteq V$.

A k-uniform hypergraph is called an ℓ-cycle if there is a cyclic ordering of the vertices such that every edge consists of k consecutive vertices, every vertex is contained in an edge and two consecutive edges (where the ordering of the edges is inherited from the ordering of the vertices) intersect in exactly ℓ-vertices. Naturally, we say that a k-uniform, n-vertex hypergraph $\mathcal{H}$ contains a Hamilton ℓ-cycle if there is a subhypergraph of $\mathcal{H}$ which forms an ℓ-cycle and which covers all vertices of $\mathcal{H}$. Note that it is necessary that $(k-\ell)$ divides n which we indicate by $n \in (k-\ell)\mathbb{N}$.

We study sufficient conditions on $\delta_{k-1}(\mathcal{H})$ for the existence of Hamilton ℓ-cycles in k-uniform hypergraphs $\mathcal{H}$. This research was initiated by G.Y. Katona and H.A. Kierstead [4]. These authors considered the case $\ell = k-1$ and such ℓ-cycles are sometimes called *tight* cycles. Katona and Kierstead proved that the condition $\delta_{k-1}(\mathcal{H}) \geqslant (1 - \frac{1}{2k})|V(\mathcal{H})| - k + 4 - \frac{5}{2k}$ implies the existence of a tight Hamilton path in a k-uniform hypergraph $\mathcal{H}$. The same authors suggested that, in fact, $\delta_{k-1}(\mathcal{H}) \geqslant (n-k+2)/2$ should suffice and they gave a matching lower bound construction. Recently, Rödl, Ruciński, and Szemerédi [12,14] answered the question of Katona and Kierstead approximately and showed the following.

Theorem 1 *(Rödl, Ruciński, and Szemerédi). For every $k \geqslant 3$ and $\gamma > 0$ there exists an n_0 such that every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geqslant (1/2+\gamma)n$ contains a Hamilton $(k-1)$-cycle.*

We focus on *loose* cycles, that is ℓ-cycles for $\ell < k/2$. In this setting an edge of an ℓ-cycle only intersects its preceding and succeeding and its following edge in the cycle. Also note that if $n \in (k-\ell)\mathbb{N}$, i.e. n is a multiple of $(k-\ell)$, then a Hamilton $(k-1)$-cycle contains a Hamilton ℓ-cycle. Consequently, the minimum degree condition for ℓ-cycles is bounded by the degree condition for $(k-1)$-cycles. The first result considering (loose) Hamilton 1-cycles for 3-uniform hypergraphs is due to Kühn and Osthus [9].

Theorem 2 *(Kühn and Osthus). For every $\gamma > 0$ there exists an n_0 such that every 3-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with n even and $\delta_2(\mathcal{H}) \geqslant (1/4+\gamma)n$ contains a Hamilton 1-cycle.*

Kühn and Osthus also showed that this result is best possible up to the error term γn (see Fact 4 below) and conjectured that $\delta_{k-1}(\mathcal{H}) \geqslant (\frac{1}{2(k-1)} + o(1))n$ should force Hamilton 1-cycles in k-uniform hypergraphs. We verify this conjecture and prove, more generally, the analogous result for ℓ-cycles with $\ell < k/2$.

Theorem 3 *(Main result). For all integers $k \geqslant 3$ and $1 \leqslant \ell < k/2$ and every $\gamma > 0$ there exists an n_0 such that every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $n \in (k-\ell)\mathbb{N}$ and $\delta_{k-1}(\mathcal{H}) \geqslant (\frac{1}{2(k-\ell)} + \gamma)n$ contains a Hamilton ℓ-cycle.*

For the case $\ell = 1$ this bound was proved independently by Keevash, Kühn, Mycroft and Osthus [6]. However, their approach uses the Blow-up lemma for hypergraphs [5] and is substantially different from ours which is based on the weak hypergraph regularity lemma, Theorem 14, and the "absorption technique" of Rödl, Ruciński, and Szemerédi introduced in [12].

The Theorem 3 is approximately best possible as the following straightforward extension of a construction from [9] shows.

Fact 4. *For every $1 \leqslant \ell < k/2$ and $n \in 2(k-\ell)\mathbb{N}$ there exists a k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n$ vertices with $\delta_{k-1}(\mathcal{H}) \geqslant \frac{n}{2(k-\ell)} - 1$, which contains no Hamilton ℓ-cycle.*

Proof. Consider the following k-uniform hypergraph $\mathcal{H} = (V, E)$. Let $A \dot\cup B = V$ be a partition of V with $|A| = \frac{n}{2(k-\ell)} - 1$ and let E be the set of all k-tuples from V with at least one vertex in A. Clearly, $\delta_{k-1}(\mathcal{H}) = |A| = \frac{n}{2(k-\ell)} - 1$. Now consider an arbitrary cycle in $\mathcal{H}$. Since $\ell < k/2$ every vertex, in particular every vertex from A, is contained in at most 2 edges of this cycle. Moreover, every edge of the cycle must intersect A. Consequently, the cycle contains at most $2|A| < n/(k-\ell)$ edges and, hence, cannot be a Hamilton cycle. $\square$

We note that the construction from Fact 4 also works in the case $\ell = k/2$ for even k. However, for that case a better construction is known. More generally, if $k - \ell$ divides k and $n \in k\mathbb{N}$, then a Hamilton ℓ-cycle contains a perfect matching. Lower and upper bounds for sufficient conditions on the minimum $(k-1)$-degree for perfect matchings were studied in [10,11,13,15]. In particular, a simple construction shows that $\delta_{k-1}(\mathcal{H}) \geqslant n/2 - k$ is necessary for perfect matchings and, hence, the same condition is required for Hamilton ℓ-cycles, if $k - \ell$ divides k. On the other hand, Theorem 1 shows that this condition is also approximately sufficient, thus, leaving only the case when k is not a multiple of $k - \ell$ and $\ell > k/2$ open.

For this case, a similar construction as given in Fact 4 combined with Theorem 1 shows that for $1 \leqslant \ell < k$ arbitrary the sufficient minimum $(k-1)$-degree condition lies between

$$\frac{n}{\lceil k/(k-\ell)\rceil (k-\ell)} \quad \text{and} \quad \left(\frac{1}{2} + o(1)\right)n.$$

Very recently it was shown by Kühn, Mycroft, and Osthus [8] that, indeed, if k is not a multiple of $(k-\ell)$, then the lower bound is approximately sufficient.

2. Proof of the main result

The proof of Theorem 3 follows the approach of Rödl, Ruciński, and Szemerédi from [12] and will be given in Section 2.3. This approach is based on three auxiliary lemmas, which we introduce in Section 2.2. We start with an outline of the proof.

2.1. Outline of the proof

We will build the Hamilton ℓ-cycles by connecting ℓ-paths. An ℓ-path (with distinguished ends) is defined similarly to ℓ-cycles. Formally, a k-uniform hypergraph $\mathcal{P}$ is an **ℓ-path** if there is an ordering $(v_0, \dots, v_{t-1})$ of its vertices such that every edge consists of k consecutive vertices and two consecutive edges intersect in exactly ℓ vertices. The ordered ℓ-sets $F^{\text{beg}} = (v_0, \dots, v_{\ell-1})$ and $F^{\text{end}} = (v_{t-\ell}, \dots, v_{t-1})$ are called the **ends of** $\mathcal{P}$. Note that this require that $t - \ell$ is a multiple of $k - \ell$. Furthermore, for loose paths (i.e. $\ell < k/2$) the ordering of the ends of an ℓ-path do not matter and we may refer to F^{beg} and F^{end} as sets.

The first lemma, the Absorbing Lemma (Lemma 5), asserts that for $\ell < k/2$ every n-vertex, k-uniform hypergraph $\mathcal{H} = (V, E)$ with $\delta_{k-1}(\mathcal{H}) \geqslant \varepsilon n$ contains a special, so-called *absorbing*, ℓ-path $\mathcal{P}$, which has the following property: For every set $U \subset V \setminus V(\mathcal{P})$ with $|U| \in (k-\ell)\mathbb{N}$ and $|U| \leqslant \alpha n$ (for some appropriate $0 < \alpha \ll \varepsilon$) there exists an ℓ-path $\mathcal{Q}$ with the same ends as $\mathcal{P}$, which covers precisely the vertices $V(\mathcal{P}) \cup U$.

The Absorbing Lemma reduces the problem of finding a Hamilton ℓ-cycle to the simpler problem of finding an almost spanning ℓ-cycle, which contains the absorbing path $\mathcal{P}$ and covers at least $(1-\alpha)n$ of the vertices. We approach this simpler problem as follows. Let $\mathcal{H}'$ be the induced subhypergraph $\mathcal{H}$, which we obtain after removing the vertices of the absorbing path $\mathcal{P}$ guaranteed by the Absorbing Lemma. We remove from $\mathcal{H}'$ a "small" set R of vertices, called *reservoir* (see Lemma 6), which has the property, that every $(k-1)$-tuple of V has "many" neighbours in R. Let $\mathcal{H}''$ be the remaining hypergraph after removing the vertices from R. Note that the property of R allows us to connect every pair $\mathcal{P}_1$ and $\mathcal{P}_2$ of disjoint ℓ-paths in $\mathcal{H}''$ to one ℓ-path, by connecting the end F_1^{end} of $\mathcal{P}_1$ with the beginning F_2^{beg} of $\mathcal{P}_2$ by one edge, where the additional $k - 2\ell$ vertices come from R.

We will choose $\mathcal{P}$ and R small enough, so that $\delta_{k-1}(\mathcal{H}'') \geqslant (\frac{1}{2(k-\ell)} + o(1))|V(\mathcal{H}'')|$. The third auxiliary lemma, the Path-cover Lemma (Lemma 7), asserts that all but $o(n)$ vertices of $\mathcal{H}''$ can be covered by a family of pairwise disjoint ℓ-paths and, moreover, the number of those paths will be constant (independent of n). Consequently, we can connect those paths and $\mathcal{P}$ to form an ℓ-cycle by using exclusively vertices from R. This way we obtain an ℓ-cycle in $\mathcal{H}$, which covers all but the $o(n)$ left-over vertices from $\mathcal{H}''$ and some left-over vertices from R. However, we will ensure that the number of those yet uncovered vertices will be smaller than αn and, hence, we can appeal to the absorption property of $\mathcal{P}$ and obtain a Hamilton ℓ-cycle.

We now state the Absorbing Lemma, the Reservoir Lemma, and the Path-cover Lemma and give the details of the outline above in Section 2.3.

2.2. Auxiliary lemmas

We start with the Absorbing Lemma. This lemma asserts the existence of a relatively "short", but powerful ℓ-path $\mathcal{P}$ which can "absorb" any small set $U \subseteq V \setminus V(\mathcal{P})$. The proof will be carried out in Section 3.

Lemma 5 *(Absorbing Lemma). For all integers $k \geqslant 3$ and $1 \leqslant \ell < k/2$ and every $\varepsilon > 0$ there exists an $\alpha > 0$ and an n_0 such that for every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geqslant \varepsilon n$ the following holds. There exists an ℓ-path $\mathcal{P} \subset \mathcal{H}$ with $|V(\mathcal{P})| \leqslant \varepsilon^5 n$ such that for all subsets $U \subset V \setminus V(\mathcal{P})$ of size at most $|U| \leqslant \alpha n$ and $|U| \in (k-\ell)\mathbb{N}$ there exists an ℓ-path $\mathcal{Q} \subset \mathcal{H}$ with $V(\mathcal{Q}) = V(\mathcal{P}) \cup U$ and, moreover, $\mathcal{P}$ and $\mathcal{Q}$ have exactly the same ends.*

The next lemma provides a reservoir $R \subset V$ which we will use to connect short paths to a long one. For a k-uniform hypergraph $\mathcal{H} = (V, E)$, a subset of the vertices $R \subseteq V$ and a $(k-1)$-tuple $S \in \binom{V}{k}$, we denote the set of neighbours of S in R by $N_R(S) = \{v \in R \setminus S\colon\ S \cup \{v\} \in E\}$ and define $\deg_R(S) = |N_R(S)|$.

Lemma 6 *(Reservoir Lemma). For every integer $k \geqslant 2$ and every reals d, $\varepsilon > 0$ there exists an n_0 such that for every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geqslant dn$ the following holds. There is a set R of size at most εn such that for all $(k-1)$-sets $S \in \binom{V}{k-1}$ we have $\deg_R(S) \geqslant d\varepsilon n/2$.*

Lemma 6 follows directly from the sharp concentration of the hypergeometric distribution.

Proof. For given k, d, and ε we choose n_0 sufficiently large and set $q = \lfloor \varepsilon n \rfloor$. From $\binom{V}{q}$, the set of all subsets of V with size q, we choose a set R uniformly at random. Now let $S \in \binom{V}{k-1}$ be an arbitrary set of size $(k-1)$ and let $X_S = |N_R(S)|$. Then X_S is hypergeometrically distributed with expectation $\mathbb{E}[X_S] \geqslant qd \geqslant 6$. Applying Chernoff's inequality for hypergeometric distribution (see, e.g., [3, Theorem 2.10]) we obtain

$$\mathbb{P}\big[X_S \leqslant \lceil dq/2 \rceil\big] \leqslant \exp(-dq/30) = \exp(-d\varepsilon n/30).$$

Thus, with probability $1 - \binom{n}{k-1}\exp(-d\varepsilon n/30) = 1 - o(1)$ every set $S \in \binom{V}{k-1}$ has at least $d\varepsilon n/2$ neighbours in R. $\square$

Finally, we state the Path-cover Lemma. By an ℓ**-path packing** of a k-uniform hypergraph $\mathcal{H}$ we mean a family of pairwise vertex disjoint ℓ-paths. Then the Path-cover Lemma asserts that a k-uniform hypergraph $\mathcal{H}$ with $\delta_{k-1}(\mathcal{H}) \geqslant (\frac{1}{2(k-\ell)} + o(1))|V(\mathcal{H})|$ can be almost perfectly covered by "few" ℓ-paths.

Lemma 7 *(Path-cover Lemma). For all integers $k \geqslant 3$ and $1 \leqslant \ell < k/2$ and every γ and $\varepsilon > 0$ there exist integers p and n_0 such that for every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geqslant (\frac{1}{2(k-\ell)} + \gamma)n$ the following holds. There is an ℓ-path packing of $\mathcal{H}$ consisting of at most p paths, which covers all but at most εn vertices of $\mathcal{H}$.*

The proof of Lemma 7 is based on the weak hypergraph regularity lemma and is given in Section 4.

2.3. Proof of Theorem 3

In this section we give the proof of the main result, Theorem 3. The proof is based on the three auxiliary lemmas introduced in Section 2.2 and follows the outline given in Section 2.1.

Proof of Theorem 3. Let integers $k \geqslant 3$ and $1 \leqslant \ell < k/2$ and a real $\gamma > 0$ be given. Applying the Absorbing Lemma (Lemma 5) for k, ℓ, and $\varepsilon_5 = \gamma/4$ we obtain $\alpha > 0$ and n_5. Next we apply the Reservoir Lemma (Lemma 6) for k, ℓ, and $d = 1/(2k)$ and $\varepsilon_6 = \min\{\alpha/2, \gamma/4\}$ we obtain n_6. Finally, we apply the Path-cover Lemma (Lemma 7) with $\gamma_7 = \gamma/2$ and $\varepsilon_7 = \alpha/2$ to obtain p and n_7. For n_0 we choose $n_0 = \max\{n_5, 2n_6, 2n_7, 16(p+1)k^2/\varepsilon_6\}$.

Now let $n \geqslant n_0$, $n \in (k-\ell)\mathbb{N}$ and let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph on n vertices with

$$\delta_{k-1}(\mathcal{H}) \geqslant \left(\frac{1}{2(k-\ell)} + \gamma\right)n.$$

Let $\mathcal{P}_0 \subset \mathcal{H}$ be the absorbing ℓ-path guaranteed by Lemma 5 (applied with k, ℓ, and ε_5). Let F_0^{beg} and F_0^{end} be the ends of $\mathcal{P}_0$ which we may refer to as sets. Note that

$$\left|V(\mathcal{P}_0)\right| \leqslant \varepsilon_5^5 n < \gamma n/4.$$

Moreover, the path $\mathcal{P}_0$ has the absorption property, i.e. for all $U \subset V \setminus V(\mathcal{P}_0)$ with $|U| \leqslant \alpha n$ and $|U| \in (k-\ell)\mathbb{N}$

$$\exists \ell\text{-path } \mathcal{Q} \subset \mathcal{H} \quad \text{s.t. } V(\mathcal{Q}) = V(\mathcal{P}_0) \cup U \text{ and } \mathcal{Q} \text{ has the ends } F_0^{\mathrm{beg}} \text{ and } F_0^{\mathrm{end}}. \tag{1}$$

Let $V' = (V \setminus V(\mathcal{P}_0)) \cup F_0^{\mathrm{beg}} \cup F_0^{\mathrm{end}}$ and let $\mathcal{H}' = \mathcal{H}[V'] = (V', E(\mathcal{H}) \cap \binom{V'}{k})$ be the induced subhypergraph of $\mathcal{H}$ on V'. Note that $\delta_{k-1}(\mathcal{H}') \geqslant (\frac{1}{2(k-\ell)} + 3\gamma/4)n \geqslant |V'|/(2k) = d|V'|$.

Due to Lemma 6 we can choose a set $R \subset V' \setminus (F_0^{\mathrm{beg}} \cup F_0^{\mathrm{end}})$ of size at most $\varepsilon_6|V'| \leqslant \varepsilon_6 n$ such that

$$\left|\deg_R(S)\right| \geqslant \varepsilon_6\left|V'\right|/(4k) - \left|F_0^{\mathrm{beg}} \cup F_0^{\mathrm{end}}\right| \geqslant \varepsilon_6 n/(8k) \quad \text{for every } S \in \binom{V'}{k-1}. \tag{2}$$

Set $V'' = V \setminus (V(\mathcal{P}_0) \cup R)$ and let $\mathcal{H}'' = \mathcal{H}[V'']$ be the induced subhypergraph of $\mathcal{H}$ on V''. Clearly,

$$\delta_{k-1}\big(\mathcal{H}''\big) \geqslant \left(\frac{1}{2(k-\ell)} + 3\gamma/4 - \varepsilon_6\right)n \geqslant \left(\frac{1}{2(k-\ell)} + \gamma/2\right)\left|V''\right|.$$

Consequently, Lemma 7 applied to $\mathcal{H}''$ (with γ_7 and ε_7) yields an ℓ-path packing of $\mathcal{H}''$ which covers all but at most $\varepsilon_7|V''| \leqslant \varepsilon_7 n$ vertices from V'' and consists of at most p paths. We denote the set of the uncovered vertices in V'' by T. Further, let $\mathcal{P}_1, \mathcal{P}_2 \ldots, \mathcal{P}_q$ with $q \leqslant p$ denote the ℓ-paths of the packing and let F_i^{beg} and F_i^{end} for $i = 1, \ldots, q$ be the ends of the ℓ-path $\mathcal{P}_i$. Recall that the ends of the absorbing ℓ-path $\mathcal{P}_0$ are F_0^{beg} and F_0^{end}. Note that for each $0 \leqslant i, j \leqslant q$ we have $|F_i^{\mathrm{end}} \cup F_j^{\mathrm{beg}}| = 2\ell < k$. Thus, for any set $X \subset R$ of size $k - 2\ell - 1$ (X might be empty) we have $\deg_R(F_i^{\mathrm{end}} \cup F_j^{\mathrm{beg}} \cup X) \geqslant \varepsilon_6 n/(8k) > (p+1)k$ due to (2) and the choice of n_0.

Consequently, for each $i \in \{0, 1, \ldots, q\}$ we can choose a set $Y_i \subset R \setminus (\bigcup_{0 \leqslant j < i} Y_j)$ such that $F_i^{\mathrm{end}} \cup Y_i \cup F_{(i+1) \bmod (q+1)}^{\mathrm{beg}}$ is an edge in $E(\mathcal{H}) \setminus \bigcup_{i=0}^{q} E(\mathcal{P}_i)$. Hence, we can connect all paths $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_q$, and $\mathcal{P}_0$ to an ℓ-cycle $\mathcal{C} \subseteq \mathcal{H}$.

Let $U = V \setminus V(\mathcal{C})$ be the set of vertices not covered by the ℓ-cycle $\mathcal{C}$. Since $U \subseteq R \cup T$ we have $|U| \leqslant (\varepsilon_7 + \varepsilon_6)n \leqslant \alpha n$. Moreover, since $\mathcal{C}$ is an ℓ-cycle and $n \in (k-\ell)\mathbb{N}$ we have $|U| \in (k-\ell)\mathbb{N}$. Thus, using the absorption property of $\mathcal{P}_0$ (see (1)) we can replace the subpath $\mathcal{P}_0$ in $\mathcal{C}$ by a path $\mathcal{Q}$ (since $\mathcal{P}_0$ and $\mathcal{Q}$ have the same ends) and since $V(\mathcal{Q}) = V(\mathcal{P}_0) \cup U$ the resulting ℓ-cycle is a Hamilton ℓ-cycle of $\mathcal{H}$. $\square$

3. Proof of the Absorbing Lemma

In this section we prove Lemma 5, the Absorbing Lemma. Roughly speaking, "absorption" stands for a local extension of a given structure, which preserves the global structure. For ℓ-paths, e.g., we want to insert a set S of vertices to an existing ℓ-path, i.e. to "absorb" S, in such a way that the new object is again an ℓ-path which, moreover, has the same ends.

Definition 8. Let $k \geqslant 3$ and $1 \leqslant \ell < k/2$ be integers and $\mathcal{H} = (V, E)$ be a k-uniform hypergraph. We say an ℓ-path with three edges $\mathcal{P} \subseteq \mathcal{H}$ and ends F^{beg} and F^{end} is an **absorbing path** for a $(k-\ell)$-set $S \in \binom{V \setminus V(\mathcal{P})}{k-\ell}$, if there exists an ℓ-path $\mathcal{Q}$ with four edges with the same ends F^{beg} and F^{end} and $V(\mathcal{Q}) = V(\mathcal{P}) \cup S$.

Moreover, if $\mathcal{P}$ is an absorbing path for S with ends F^{beg} and F^{end}, then we call the t-set $T = V(\mathcal{P}) \in \binom{V \setminus S}{t}$ with $t = 3(k-\ell)+\ell$ an **absorbing t-tuple for** S with ends F^{beg} and F^{end}.

Given that an absorbing ℓ-path $\mathcal{P}$ for S was part of some long ℓ-path, then the local change of absorbing S does not destroy the long path since the ends of $\mathcal{P}$ and $\mathcal{Q}$ are the same. Clearly, for any fixed $(k-\ell)$-set S there are at most $O(n^t)$ absorbing t-tuples. The following proposition says that this bound is achieved up to a constant factor when the minimum $(k-1)$-degree of $\mathcal{H}$ is linear in n.

Proposition 9. *Let $k \geqslant 3$, $1 \leqslant \ell < k/2$, $\varepsilon > 0$, and let $\mathcal{H}$ be a k-uniform hypergraph on $n \geqslant 6k/\varepsilon$ vertices with $\delta_{k-1}(\mathcal{H}) \geqslant \varepsilon n$. Then for every $(k-\ell)$-set $S \in \binom{V}{k-\ell}$ there are at least $\varepsilon^5 \binom{n}{t}/(2^{5+3k}k^4)$ absorbing t-tuples $T \in \binom{V \setminus S}{t}$ with $t = 3(k-\ell)+\ell$.*

We postpone the proof of Proposition 9 and we first deduce Lemma 5 from it.

Proof of Lemma 5. Let $k \geqslant 3$, $1 \leqslant \ell < k/2$, and $\varepsilon > 0$ be given. We set $t = 3(k-\ell)+\ell$ and fix auxiliary constants

$$\zeta = \frac{\varepsilon^5 (t-2\ell)!}{2^{6+3k} k^4 t!} \quad \text{and} \quad \varrho = \frac{\zeta}{16t^2} < \frac{\varepsilon^5}{8t}.$$

Finally we set

$$\alpha = \zeta \varrho / 4$$

and let $n_0 \geqslant 6k/\varepsilon$ be sufficiently large.

Suppose $\mathcal{H} = (V, E)$ is a k-uniform hypergraph on $n \geqslant n_0$ vertices which satisfies $\delta_{k-1}(\mathcal{H}) \geqslant \varepsilon n$. Note that in Proposition 9 the ends of the absorbing t-tuples are not specified yet. This we do retrospectively by taking the ends $F_T^{\mathrm{beg}}, F_T^{\mathrm{end}} \subset T$ of an arbitrary t-set $T \in \binom{V}{t}$ uniformly at random, i.e. with probability $(t-2\ell)!/t!$ a given pair of disjoint, ordered ℓ-tuples will become the ends of T. Hence, due to Proposition 9, the expected number of absorbing t-tuples (now with distinguished ends) for a fixed $(k-\ell)$-set $S \in \binom{V}{k-\ell}$ is at least $2\zeta \binom{n}{t}$. Applying Chernoff's inequality we derive that there is a choice of ends for all t-sets which yields at least $\zeta \binom{n}{t}$ absorbing t-tuples with distinguished ends for all $(k-\ell)$-sets. We fix such a choice and for a fixed $(k-\ell)$-set $S \in \binom{V}{k-\ell}$ let $\mathcal{T}(S)$ denote the set of the absorbing t-tuples T for S with ends F_T^{beg} and F_T^{end} according to this choice. Thus, we have $|\mathcal{T}(S)| \geqslant \zeta \binom{n}{t}$ for all $S \in \binom{V}{k-\ell}$.

Next we pick a family $\mathscr{T} \subseteq \binom{V}{t}$ randomly, where each t-tuple $T \in \binom{V}{t}$ is included in $\mathscr{T}$ independently with probability $p = \varrho n / \binom{n}{t}$. Hence, we have

$$\mathbb{E}\big[|\mathscr{T}|\big] = \varrho n \quad \text{and} \quad \mathbb{E}\big[|\mathscr{T} \cap \mathcal{T}(S)|\big] \geqslant \zeta \varrho n, \quad S \in \binom{V}{k-\ell}.$$

From Chernoff's inequality we infer that with probability $1 - o(1)$

$$|\mathscr{T}| \leqslant 2\varrho n \tag{3}$$

and

$$\left|\mathscr{T} \cap \mathcal{T}(S)\right| \geqslant \zeta\varrho n/2 \quad \text{for all } S \in \binom{V}{k-\ell}. \tag{4}$$

Furthermore, let $I(\mathscr{T})$ denote the number of intersecting t-tuples in $\mathscr{T}$, i.e. the number of pairs T and $T' \in \mathscr{T}$ such that $T \cap T' \neq \emptyset$. Then

$$\mathbb{E}\big[I(\mathscr{T})\big] \leqslant t\binom{n}{t}\binom{n}{t-1} \times p^2 = \frac{t^2\varrho^2 n^2}{n-t+1} \leqslant 2t^2\varrho^2 n = \zeta\varrho n/8$$

due to the choice of ϱ, and using Markov's inequality we conclude that with probability at least 1/2

$$I(\mathscr{T}) \leqslant \zeta\varrho n/4. \tag{5}$$

In particular, the properties (3), (4), and (5) hold simultaneously with positive probability for the randomly chosen family $\mathscr{T}$. So, let $\mathscr{T}'$ be a family satisfying (3), (4), and (5). By deleting all intersecting t-tuples from $\mathscr{T}'$ and all those t-tuples which do not absorb any $S \in \binom{V}{k-\ell}$ we obtain a family $\mathscr{T}'' \subset \mathscr{T}'$ of pairwise disjoint t-tuples of size at most $2\varrho n$ which, due to (4), (5), and the choice of α, satisfies

$$\left|\mathscr{T}'' \cap \mathcal{T}(S)\right| \geqslant \zeta\varrho n/4 = \alpha n \tag{6}$$

for all $S \in \binom{V}{k-\ell}$.

Lastly, we want to connect the t-tuples in $\mathscr{T}''$ to create an ℓ-path. To this end, let $\mathscr{T}'' = \{T_1, \ldots, T_r\}$ for some $r \leqslant 2\varrho n$ and let F_i^{beg} and F_i^{end} be the ends of T_i. Since every T_i (with its chosen ends F_i^{beg} and F_i^{end}) absorbs at least one $(k-\ell)$-set, the induced hypergraph $\mathcal{H}[T_i]$ must contain an ℓ-path $\mathcal{P}_i$ with three edges and ends F_i^{beg} and F_i^{end}. For $i = 1, \ldots, r-1$ observe further that $|F_i^{\text{end}} \cup F_{i+1}^{\text{beg}}| = 2\ell$ and, hence, for any V_i of size at least $n - 4\varrho n t$ and any $Y \in \binom{V_i}{k-2\ell-1}$ we know

$$\left|N_{V_i}\big(F_i^{\text{end}} \cup F_{i+1}^{\text{beg}} \cup Y\big)\right| \geqslant \varepsilon n - 4\varrho n t > 0.$$

Thus, we can choose $X_i \in N_{V_i}(F_i^{\text{end}} \cup F_{i+1}^{\text{beg}})$ to connect $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ through the edge $F_i^{\text{end}} \cup X_i \cup F_{i+1}^{\text{beg}}$. Starting with the set $V_1 = V(\mathcal{H}) \setminus \bigcup_{T \in \mathscr{T}''} V(T)$ of size $|V_1| \geqslant n - 2\varrho n t$ we connect $\mathcal{P}_1$ and $\mathcal{P}_2$. We continue by induction. So suppose for some $i < r$ we chose sets $X_1, \ldots, X_{i-1}$ and used them to connect the ℓ-paths $\mathcal{P}_1, \ldots, \mathcal{P}_i$ to one ℓ-path. With $V_i = V_1 \setminus (\bigcup_{j=1}^{i-1} X_j)$ which has size at least $n - 2\varrho n t - i(k-2\ell) > n - 4\varrho n t$ and by the observation from above we connect $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ by choosing $X_i \in N_{V_i}(F_i^{\text{end}} \cup F_{i+1}^{\text{beg}})$. Consequently, we can connect all ℓ-paths $\mathcal{P}_1, \ldots, \mathcal{P}_r$ to one ℓ-path $\mathcal{P}$ containing at most $4\varrho n t \leqslant \varepsilon^5 n$ vertices.

Finally, suppose $U \subset V \setminus V(\mathcal{P})$ with $|U| \leqslant \alpha n$ and $|U| \in (k-\ell)\mathbb{N}$. Then we partition U into $q \leqslant \alpha n/(k-\ell)$ pairwise disjoint sets $S_1, \ldots, S_q$ each of size $(k-\ell)$. But since (6) holds, we can absorb each S_i, $i = 1, \ldots, q$ one by one taking an unused absorbing t-tuple $T_i \in \mathscr{T}'' \cap \mathscr{T}_S$ for each S_i. This way we obtain an ℓ-path $\mathcal{Q}$ which covers exactly the vertices in $V(\mathcal{P}) \cup U$ and the lemma follows. $\square$

We complete the proof of Lemma 5 by proving Proposition 9. To this end we need the notion of a "neighbourhood" of a set $S \subset V(\mathcal{H})$ in a set $U \subset V(\mathcal{H})$. This is given by $N_U(S) = \{X \subset U \setminus S\colon\ S \cup X \in E(\mathcal{H})\}$.

Proof of Proposition 9. Let $S \in \binom{V}{k-\ell}$ be an arbitrary set of size $k-\ell$ and set $V_0 = V \setminus S$. In the following we will choose pairwise disjoint sets A, B_1, B_2, C, D_1, and D_2 whose union forms an absorbing t-tuple for S.

We start by choosing $A \in \binom{V_0}{k-2\ell}$ arbitrarily. Then the number of choices for A is

$$\binom{n-k+\ell}{k-2\ell}. \tag{7}$$

Set $V_1 = V_0 \setminus A$ and split $S \dot\cup A = Z_1 \dot\cup L \dot\cup Z_2$ in an arbitrary way such that $|L| = \ell$ and $|Z_1| = |Z_2| = k-2\ell$. We choose $B_1 \in N_{V_1}(Z_1 \cup L)$ and $B_2 \in N_{V_2}(Z_2 \cup L)$ where $V_2 = V_1 \setminus B_1$. To compute the number of choices for B_1 and B_2 note that $|V_2| = n-2k+3\ell$, $|V_3| = n-2k+2\ell$ and for every set $X_i \in \binom{V_i}{\ell-1}$, $i=1,2$, we know that $\deg_{\mathcal{H}}(Z_i \cup L \cup X_i) \geqslant \varepsilon n$ thus $N_{V_i}(Z_i \cup L \cup X_i)$ has size at least $\varepsilon n - 2k \geqslant \varepsilon n/2$, since $n \geqslant 4k/\varepsilon$. This way we count each possible B_i in ℓ ways. Consequently, the number of choices for B_1 and B_2, i.e. $|N_{V_2}(Z_1 \cup L)| \times |N_{V_3}(Z_2 \cup L)|$ is at least

$$\left(\frac{\varepsilon n}{2\ell}\right)^2 \binom{n-2k+3\ell}{\ell-1}\binom{n-2k+2\ell}{\ell-1}. \tag{8}$$

Next, set $V_3 = V_2 \setminus B_2$ and for $i=1,2$ let $B_i' \subset B_i$ of size $|B_i'| = |B_i| - 1$ (thus, B_i' may be empty if $\ell = 1$). We choose the set $C \in N_{V_3}(A \cup B_1' \cup B_2')$. Since $|V_3| = n-2k+\ell$ by arguing as above for B_1 and B_2 we conclude that the number of choices for C is at least

$$\frac{1}{2}(n-2k+\ell)(\varepsilon n - 2k) \geqslant \frac{\varepsilon n^2}{8}. \tag{9}$$

Then we set $V_4 = V_5 \setminus C$ and for $C = \{v_1, v_2\}$, we choose $D_1 \in N_{V_4}(B_1 \cup \{v_1\})$ and with $V_5 = V_4 \setminus D_1$ we choose $D_2 \in N_{V_5}(B_2 \cup \{v_2\})$. Note that $|V_5| = n-2k+\ell-2$, $|V_6| = n-3k-1$ and $|B_i \cup \{v_i\}| = \ell+1$. Thus, again, by arguing as for B_1, B_2 we derive that the number of choices for D_1 and D_2 is at least

$$\left(\frac{\varepsilon n}{2(k-\ell-1)}\right)^2 \binom{n-2k+\ell-2}{k-\ell-2}\binom{n-3k-1}{k-\ell-2}. \tag{10}$$

For given S let

$$T = A \dot\cup B_1 \dot\cup B_2 \dot\cup C \dot\cup D_1 \dot\cup D_2$$

and note that

$$|T| = |A| + |B_1| + |B_2| + |C| + |D_1| + D_2| = 3(k-\ell) + \ell = t.$$

Combining (7), (8), (9), and (10) we obtain that the number of choices for T chosen as above for a given set S is at least

$$\frac{\varepsilon^5}{2^7 \ell^2 k^2}\binom{n-k+\ell}{t} \geqslant \frac{\varepsilon^5}{2^{7+t}\ell^2 k^2}\binom{n}{t} \geqslant \frac{\varepsilon^5}{2^{5+3k}k^4}\binom{n}{t}.$$

We now verify that T is indeed an absorbing t-tuple for S. For that we “reorder” the vertices of T and observe that

$$T = D_1 \dot\cup B_1 \dot\cup \{v_1\} \dot\cup A \dot\cup \{v_2\} \dot\cup B_2 \dot\cup D_2.$$

Note that

$$E_1 = D_1 \dot\cup B_1 \dot\cup \{v_1\}, \qquad G = B_1' \dot\cup \{v_1\} \dot\cup A \dot\cup \{v_2\} \dot\cup B_2', \quad \text{and} \quad E_2 = \{v_2\} \dot\cup B_2 \dot\cup D_2$$

are edges in $\mathcal{H}$ and form an ℓ-path $\mathcal{P}$ with three edges, since $|E_i \cap G| = |B_i' \cup \{v_i\}| = \ell$, for $i = 1,2$. For the ends of this path we could fix any ordering of any ℓ-set from D_i. Moreover, the sets

$$G_1 = B_1 \dot\cup Z_1 \dot\cup L \quad \text{and} \quad G_2 = L \dot\cup Z_2 \dot\cup B_2$$

are also edges of $\mathcal{H}$ and E_1, G_1, G_2, E_2 forms an ℓ-path $\mathcal{Q}$ with $V(\mathcal{Q}) = S \dot\cup T$, since $|G_i \cap E_i| = |B_i| = \ell$, for $i = 1,2$ and $|G_1 \cap G_2| = |L| = \ell$. The ends of this ℓ-path can be chosen to coincide with the ends of $\mathcal{P}$, since $D_i \cap G_i = \emptyset$ for $i = 1,2$.

This proves that any set T chosen as above is indeed an absorbing t-tuple for the set S. $\square$

4. The Path-cover Lemma

In this section we prove the Path-cover Lemma, Lemma 7. The proof combines the techniques in [14] and [9] and relies on the so called *weak hypergraph regularity lemma*, a straightforward generalisation of Szemerédi's regularity lemma [17] for graphs (see e.g. [1,2,16]).

4.1. Almost perfect $\mathcal{F}_{k,\ell}$-packings

First we show that an n-vertex, k-uniform hypergraph $\mathcal{H}$ with minimum degree $\delta_{k-1}(\mathcal{H}) \geqslant n/(2(k-\ell))$ contains a $\mathcal{F}_{k,\ell}$-packing which covers all but $o(n)$ vertices of $\mathcal{H}$, where $\mathcal{F}_{k,\ell}$ is defined as follows.

Definition 10. For positive integers k and ℓ let $\mathcal{F}_{k,\ell}$ be the k-uniform hypergraph on $2(k-\ell)(k-1)$ vertices whose vertex set falls into pairwise disjoint sets $A_1, A_2, \dots, A_{2k-2\ell-1}, B$ each of size $k-1$ and whose edge set consists of all sets $A_i \cup \{b\}$ where $i \in [2k-2\ell-1]$ and $b \in B$.

Kühn and Osthus [9] considered $\mathcal{F}_{3,1}$-packings, i.e. families of pairwise vertex disjoint copies of $\mathcal{F}_{3,1}$. The proof of the $\mathcal{F}_{k,\ell}$-packing lemma, Lemma 11, follows their approach.

Lemma 11 *($F_{k,\ell}$-packing lemma). For all integers $k \geqslant 3$ and $1 \leqslant \ell < k$ and every $\varepsilon > 0$ there exists an n_0 such that for every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices the following holds.*

If $\deg_{k-1}(S) \geqslant n/(2(k-\ell))$ for all but at most εn^{k-1} sets $S \in \binom{V}{k-1}$, then $\mathcal{H}$ contains a $F_{k,\ell}$-packing covering all but at most $(5\varepsilon)^{1/(k-1)}n$ vertices.

Proof. For given k, ℓ, and ε we choose n_0 large enough. Further set $\delta = (5\varepsilon)^{1/(k-1)}$. Suppose $\mathcal{A} = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{i_0}\}$ is a largest $\mathcal{F}_{k,\ell}$-packing leaving the vertex set $X \subset V$ of size $|X| \geqslant \delta n$ uncovered.

From the condition on the degree for $\mathcal{H}$ we first show the following.

Claim 12. *There is a family $\mathcal{B}$ of size $\delta n/(2k^k)$ consisting of mutually disjoint $(k-1)$-sets $S \in \binom{X}{k-1}$ such that $\deg(S) \geqslant n/(2(k-\ell))$ and $|N_X(S)| \leqslant \delta n/(4k)$ for all $S \in \mathcal{B}$.*

Proof. The claim follows from a probabilistic argument. First we split X into two parts $X = X_1 \dot\cup X_2$ by choosing $X_2 \subset X$ of size $|X|/(2k)$ uniformly at random. Thereafter, we take a family $\mathcal{S}$ consisting of $\delta n/k^k$ pairwise disjoint sets $S \in \binom{X_1}{k-1}$ from X_1 such that $\deg(S) \geqslant n/(2(k-\ell))$. Such a family exists indeed, since the number of $(k-1)$-sets with degree falling below $n/(2(k-\ell))$ is at most εn^{k-1} and due to the choice of δ

$$\binom{|X_1|}{k-1} - \varepsilon n^{k-1} \geqslant (k-1)\frac{\delta n}{k^k}\binom{|X_1|}{k-2}.$$

Next, we claim that at least half, i.e. $\delta n/(2k^k)$, of the chosen $(k-1)$-sets S_i must satisfy $|N_X(S_i)| \leqslant \delta n/(4k)$ since otherwise the $\mathcal{F}_{k,\ell}$-packing $\mathcal{A}$ was not largest possible. For a contradiction, let $\mathcal{S}' \subset \mathcal{S}$ denote the set of the chosen $S_i \in \mathcal{S}$ such that $|N_X(S)| > \delta n/(4k)$ and suppose $\mathcal{S}' = \{S_1, \dots, S_r\}$ has size $r \geqslant \delta n/(2k^k)$.

For any $(k-1)$-sets $S \in \binom{X_1}{k-1}$ with $|N_X(S)| > |X|/(4k)$ let $Y_S = |N_{X_2}(S)|$ denote the size of its neighbourhood in X_2. Then Y_S has hypergeometric distribution with mean $\mathbb{E}[Y_S] \geqslant (|X|/(4k)) \times (1/(2k)) \geqslant \delta n/(8k^2)$ and applying Chernoff's inequality we conclude

$$p = \mathbb{P}\big[|Y_S| \leqslant \delta n/\big(16k^2\big)\big] \leqslant \exp\big\{-\delta n/\big(100k^2\big)\big\}.$$

Thus, with a probability at least $1 - \binom{|X|}{k-1}p = 1 - o(1)$ all sets $S \in \binom{X}{k-1}$ with $|N_X(S)| > |X|/(4k)$ also satisfy $|N_{X_2}(S)| \geqslant n/(16k^2)$. In particular, almost surely $|N_{X_2}(S)| \geqslant n/(16k^2)$ is satisfied for all $S \in \mathcal{S}'$ and we assume that this indeed happens for the decomposition $X = X_1 \dot\cup X_2$ we have chosen. Now consider the auxiliary bipartite graph G with vertex classes $\mathcal{S}'$ and X_2 and with $\{S, v\}$ being an edge if and only if $S \cup \{v\} \in \mathcal{H}$. Then every S has at least $\delta n/(16k^2)$ neighbours, thus, by the well known result of Kövari, Turán, and Sós [7] the graph G contains a $K_{k,k-1}$. However, this $K_{k,k-1}$ in G corresponds to a copy of $\mathcal{F}_{k,\ell}$ in $\mathcal{H}$, which is a contradiction to $\mathcal{A}$ being the largest $\mathcal{F}_{k,\ell}$-packing. $\square$

Continuing the proof of Lemma 11, we fix a family $\mathcal{B} = \{S_1, \dots, S_q\}$, $q = \delta n/(2k^k)$ as stated in the claim above. For a set $S_i \in \mathcal{B}$ we say that an element $\mathcal{F}$ from the $\mathcal{F}_{k,\ell}$-packing $\mathcal{A}$ is *good* for S_i if $\mathcal{F}$ contains at least k neighbours of S_i, i.e. $|N_{V(\mathcal{F})}(S_i)| \geqslant k$. With n_i denoting the number of good $\mathcal{F} \in \mathcal{A}$ for S_i and $t = 2(k-\ell)(k-1)$ we conclude from the condition on $\deg_{k-1}(S_i)$ that

$$\frac{n}{2(k-\ell)} \leqslant \deg_{k-1}(S_i) \leqslant (k-1)\frac{(1-\delta)n}{t} + tn_i + \frac{\delta n}{4k} \tag{11}$$

$$\leqslant \frac{(1-\delta/2)n}{2(k-\ell)} + tn_i. \tag{12}$$

From this we infer that $n_i \geqslant \delta n/(8k^3) = n^*$. Next, we want to count all those pairs $(S, \mathcal{T})$ with $\mathcal{T} = \{\mathcal{F}^1, \dots, \mathcal{F}^{k-1}\} \in \binom{\mathcal{A}}{k-1}$ such that each $\mathcal{F} \in \mathcal{T}$ is good for $S \in \mathcal{B}$. Such a pair $(S, \mathcal{T})$ we call a *good pair* and the number of good pairs is at least $|\mathcal{B}|\binom{n^*}{k-1} \geqslant (\delta n)^k/(8k^5)^k$. Thus by averaging we infer that there must be a $\mathcal{T}$ and at least $\delta^k n/(8k^5)^k$ sets $S_i \in \mathcal{B}$ such that $(S_i, \mathcal{T})$ are a good pairs.

Hence, it exists a family $\mathcal{B}' \subseteq \mathcal{B}$ containing at least $(\delta^k n/(8k^5)^k)/\binom{2(k-\ell)(k-1)}{k}^{k-1}$ pairwise disjoint $(k-1)$-sets S from $\mathcal{B}$ and for every $j = 1, \dots k-1$ there exist k vertices $v_1^j, \dots, v_k^j$ in $\mathcal{F}^j$ such that

$$S \cup \{v_1^j\}, \dots, S \cup \{v_k^j\} \in E(\mathcal{H}) \quad \text{for every } S \in \mathcal{B}' \text{ and } j = 1, \dots, k-1.$$

Since $(\delta^k n/(8k^5)^k)/\binom{2(k-\ell)(k-1)}{k}^{k-1} \geqslant (2(k-\ell)-1)k$ for sufficiently large n, we can select k families mutually disjoint families $\{S_1^i, \dots, S_{2k-2\ell-1}^i\} \subseteq \mathcal{B}'$ for $i = 1, \dots, k$. Now for every $i = 1, \dots, k$ the set

$$\{S_p^i \cup \{v_i^j\} \colon p = 1, \dots, 2k-2\ell-1,\ j = 1, \dots, k-1\}$$

is the edge set of a copy of $\mathcal{F}_{k,\ell}$ and we obtain k mutually disjoint copies of $\mathcal{F}_{k,\ell}$ this way. Replacing the $(k-1)$-copies $\mathcal{F}^1, \dots, \mathcal{F}^{k-1}$ by those k copies enlarges the $\mathcal{F}_{k,\ell}$-packing $\mathcal{B}$, which is a contradiction. $\square$

4.2. *Weak hypergraph regularity and path embeddings*

In this section we introduce the so-called *weak hypergraph regularity lemma*, a straightforward extension of Szemerédi's regularity lemma [17] for graphs. Further we will find almost perfect path packings in regular k-tuples. Similar results were used by Rödl, Ruciński and Szemerédi in [14].

4.2.1. *The weak regularity lemma for hypergraphs*

Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph and let $A_1, \dots, A_k$ be mutually disjoint non-empty subsets of V. We define $e_{\mathcal{H}}(A_1, \dots, A_k)$ to be the number of edges with one vertex in each A_i, $i \in [k]$ and the **density** of $\mathcal{H}$ with respect to $(A_1, \dots, A_k)$ as

$$d_{\mathcal{H}}(A_1, \dots, A_k) = \frac{e_{\mathcal{H}}(A_1, \dots, A_k)}{|A_1| \cdot \ldots \cdot |A_k|}.$$

We say the k-tuple $(V_1, \dots, V_k)$ of mutually disjoint subsets $V_1, \dots, V_k \subseteq V$ is (ε, d)-**regular**, for constants $\varepsilon > 0$ and $d \geqslant 0$, if

$$\left| d_{\mathcal{H}}(A_1, \dots, A_k) - d \right| \leqslant \varepsilon$$

for all k-tuples of subsets $A_1 \subset V_1, \dots, A_k \subset V_k$ satisfying $|A_1| \geqslant \varepsilon |V_1|, \dots, |A_k| \geqslant \varepsilon |V_k|$. We say the k-tuple $(V_1, \dots, V_k)$ is ε-**regular** if it is (ε, d)-regular for some $d \geqslant 0$. The following fact is a direct consequence of the definition above.

Fact 13. *For an (ε, d)-regular tuple $(V_1, \ldots, V_k)$ we have*

(i) *$(V_1, \ldots, V_k)$ is (ε', d)-regular for all $\varepsilon' > \varepsilon$ and*
(ii) *if for all $i \in [k]$ the set $V_i' \subset V_i$ has size $|V_i'| \geqslant c|V_i|$, then $(V_1', \ldots, V_k')$ is $(\varepsilon/c, d)$-regular.*

As a straightforward generalisation of the original regularity lemma we obtain the following regularity lemma for graphs (see, e.g., [1,2,16]).

Theorem 14 *(Weak regularity lemma for hypergraphs). For all integers $k \geqslant 2$ and $t_0 \geqslant 1$, and every $\varepsilon > 0$, there exist $T_0 = T_0(k, t_0, \varepsilon)$ and $n_0 = n_0(k, t_0, \varepsilon)$ so that for every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $n \geqslant n_0$ vertices, there exists a partition $V = V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_t$ such that*

(i) *$t_0 \leqslant t \leqslant T_0$,*
(ii) *$|V_1| = |V_2| = \cdots = |V_t|$ and $|V_0| \leqslant \varepsilon n$,*
(iii) *for all but at most $\varepsilon\binom{t}{k}$ sets $\{i_1, \ldots, i_k\} \in \binom{[t]}{k}$, the k-tuple $(V_{i_1}, \ldots, V_{i_k})$ is ε-regular.*

A partition as given in Theorem 14 is called an **ε-regular partition** of $\mathcal{H}$ (with lower bound t_0 on the number of vertex classes). Further, we need the notion of the cluster graph.

Definition 15. For an ε-regular partition of $\mathcal{H}$ and $d \geqslant 0$ we refer to the sets $V_i, i \in [t]$ as **clusters** and define the **cluster hypergraph** $\mathcal{K} = \mathcal{K}(\varepsilon, d)$ with vertex set $[t] = \{1, 2, \ldots, t\}$ and $\{i_1, \ldots, i_k\} \in \binom{[t]}{k}$ being an edge if and only if $(V_{i_1}, \ldots, V_{i_k})$ is ε-regular and $d(V_{i_1}, \ldots, V_{i_k}) \geqslant d$.

The following proposition relates the degree condition of $\mathcal{H}$ and its cluster hypergraph $\mathcal{K}$. It shows that $\mathcal{K}$ "almost inherits" the minimum degree of $\mathcal{H}$. Let $\deg_{\mathcal{K}}(S)$ denote the degree of the set S in the cluster hypergraph $\mathcal{K}$.

Proposition 16. *Given a k-uniform hypergraph $\mathcal{H} = (V, E)$ with minimum $(k-1)$-degree*

$$\delta_{k-1}(\mathcal{H}) \geqslant \left(\frac{1}{2(k-\ell)} + \gamma\right)n$$

and an ε-regular partition $V = V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_t$ with $0 < \varepsilon < \gamma^2/16$ and $t_0 \geqslant 8k/\varepsilon \geqslant 3k/\gamma$. Further, let $\mathcal{K} = \mathcal{K}(\varepsilon, \gamma/6)$ be the cluster hypergraph of $\mathcal{H}$. Then the number of $(k-1)$-sets $S = \{i_1, \ldots, i_k\} \in \binom{[t]}{k-1}$ violating

$$\deg_{\mathcal{K}}(S) \geqslant \left(\frac{1}{2(k-\ell)} + \frac{\gamma}{4}\right)t$$

is at most $\sqrt{\varepsilon}t^{k-1}$.

Proof. Note first that the cluster hypergraph $\mathcal{K}(\varepsilon, \gamma/6)$ can be written as the intersection of two hypergraphs $\mathcal{D} = \mathcal{D}(\gamma/6)$ and $\mathcal{R} = \mathcal{R}(\varepsilon)$ both defined on the vertex set $[t]$ and

- $\mathcal{D}(\gamma/6)$ consists of all sets $\{i_1, \ldots, i_k\}$ such that $d(V_{i_1}, \ldots, V_{i_k}) \geqslant \gamma/6$,
- $\mathcal{R}(\varepsilon)$ consists of all sets $\{i_1, \ldots, i_k\}$ such that $(V_{i_1}, \ldots, V_{i_k})$ is ε-regular.

Given an arbitrary set $S \in \binom{[t]}{k-1}$ we first show

$$\deg_{\mathcal{D}}(S) \geqslant \left(\frac{1}{2(k-\ell)} + \frac{\gamma}{2}\right)t. \tag{13}$$

To this end note that $S = \{i_1, \dots, i_{k-1}\}$ represents the tuple $(V_{i_1}, \dots, V_{i_{k-1}})$ with $n/t \geqslant m := |V_{i_j}| \geqslant (1-\varepsilon)n/t$ for all $j \in [k-1]$. We consider now the number of edges in $\mathcal{H}$ which intersects each V_{i_j} in exactly one vertex. From the condition on $\delta_{k-1}(\mathcal{H})$ this is at least

$$m^{k-1}\left(\left(\frac{1}{2(k-\ell)} + \gamma\right)n - (k-1)m\right) \geqslant m^{k-1}\left(\frac{1}{2(k-\ell)} + \frac{2\gamma}{3}\right)n \tag{14}$$

since $t \geqslant t_0 \geqslant 3k/\gamma$.

On the other hand, in case (13) does not hold the same number can be bounded from above by

$$\left(\frac{1}{2(k-\ell)} + \frac{\gamma}{2}\right)t \times m^k + t \times \frac{\gamma}{6}m^k$$

with contradiction to (14).

Next, observe that there are at most $\varepsilon\binom{t}{k} < \varepsilon t^k/k$ sets $\{i_1, \dots, i_k\} \in \binom{[t]}{k}$ such that the corresponding tuples $(V_{i_1}, \dots, V_{i_k})$ are not ε-regular, i.e. $\{i_1, \dots, i_k\} \notin \mathcal{R}$. Thus, all but at most $\sqrt{\varepsilon}t^{k-1}$ sets $S \in \binom{[t]}{k-1}$ satisfy

$$\deg_{\mathcal{R}}(S) \geqslant \left(1 - \sqrt{\varepsilon}\right)t. \tag{15}$$

Since $\mathcal{K} = \mathcal{D} \cap \mathcal{R}$ the proposition follows from (13), (15) and $\sqrt{\varepsilon}t \leqslant \gamma t/4$. $\square$

4.2.2. Almost perfect path-packings in regular k-tuples

In this section we show that (ε, d)-regular k-tuples $(V_1, \dots, V_k)$ can be almost perfectly covered by ℓ-paths.

Definition 17. Suppose $\mathcal{H}$ is a k-uniform, k-partite hypergraph with partition classes $V_1, V_2, \dots, V_k$. Then we call an ℓ-path $\mathcal{P} \subset \mathcal{H}$ with t edges $(E_1, \dots, E_t)$ **canonical** with respect to $(V_1, V_2, \dots, V_k)$ if

$$E_i \cap E_{i+1} \subset \bigcup_{j \in [\ell]} V_j \quad \text{or} \quad E_i \cap E_{i+1} \subset \bigcup_{j \in [k]\setminus[k-\ell]} V_j$$

for all $i = 1, 2, \dots, t-1$.

Further, we say that V_i is in **end position** if it is one of the first or the last ℓ elements in the ordering, i.e. $i \in [\ell] \cup \{k-\ell+1, \dots, k\}$, whereas V_i is in **middle position** if $i \in \{\ell+1, \dots, k-\ell\}$.

Remark 18. Let t be an odd number. If $\mathcal{P}$ with t edges is a canonical path with respect to $(V_1, \dots, V_k)$ and $n_i = |V(\mathcal{P}) \cap V_i|$, then

$$n_i = \begin{cases} (t+1)/2 & \text{if } V_i \text{ is in end position,} \\ t & \text{if } V_i \text{ is in middle position.} \end{cases}$$

The following proposition was essentially proved in [14].

Proposition 19. *Suppose $\mathcal{H}$ is a k-partite, k-uniform hypergraph with the partition classes $V_1, V_2, \dots, V_k$, $|V_i| = m$ for all $i \in [k]$, and $|E(\mathcal{H})| \geqslant dm^k$. Then there exists a canonical ℓ-path in $\mathcal{H}$ with respect to $(V_1, \dots V_k)$ with $t > dm/(2(k-\ell))$ edges.*

Proof. First we consider all possible ends of a canonical ℓ-path $\mathcal{P}$, i.e. all ℓ-sets $L \subset V(\mathcal{H})$ such that

$$|L \cap V_i| = 1 \quad \text{either for all } i \in [\ell] \text{ or for all } i \in [k] \setminus [k-\ell].$$

For a possible end L such that $\deg(L) = |\{E \in \mathcal{H}\colon\ L \subset E\}| < dm^{k-\ell}/2$ we delete all edges from the current hypergraph which contain L. We keep doing this until every possible end L satisfies $\deg(L) = 0$ or $\deg(L) \geqslant dm^{k-\ell}/2$ in the present hypergraph. Note that we have deleted less than $2m^\ell \times dm^{k-\ell}/2 = dm^k$ edges, hence, the final hypergraph $\mathcal{H}'$ is non-empty. We pick a maximal canonical ℓ-path $\mathcal{P} \subset \mathcal{H}'$ with respect to $(V_1, \dots, V_k)$ which has $t \geqslant 1$ edges and let the ℓ-set L denote one end of $\mathcal{P}$. Since L is contained in an edge in $\mathcal{H}'$ we know that $\deg(L) \geqslant dm^{k-\ell}/2$. On the other hand, every edge in $\mathcal{H}'$ which contains L must intersect $V(\mathcal{P}) \setminus L$ since $\mathcal{P}$ is maximal. Thus, we have

$$\frac{dm^{k-\ell}}{2} \leqslant \deg(L) < \left((k-2\ell)t + \ell\frac{(t+1)}{2}\right)m^{k-\ell-1} \leqslant (k-\ell)tm^{k-\ell-1}.$$

This yields $t > dm/(2(k-\ell))$. $\square$

We want to use Proposition 19 to cover a ε-regular tuple $(V_1, \ldots, V_k)$ by ℓ-paths which intersect $V_1, \ldots, V_{k-1}$ equally and which, moreover, intersect V_k almost as little as possible.

Lemma 20. *For all integers $k \geqslant 3$, $1 \leqslant \ell < k/2$, and all $d, \beta > 0$ there exist $\varepsilon > 0$, integers p and m_0 such that for all $m > m_0$ the following holds. Suppose $\mathcal{V} = (V_1, V_2, \ldots, V_k)$ is an (ε, d)-regular k-tuple with $|V_i| = (2k-2\ell-1)m$ for all $i \in [k-1]$ and $|V_k| = (k-1)m$. Then there is a family consisting of at most p pairwise vertex disjoint ℓ-paths which cover all but at most βm vertices of $\mathcal{V}$.*

Proof. Let k, ℓ, d, and β be given. We choose $\varepsilon = \min\{d/2, \beta/(7k^2), 1/k!\}$, $p = 2k/\varepsilon^2$, and $m_0 > 2\varepsilon^{-3}$ sufficiently large. Suppose $\mathcal{V} = (V_1, \ldots, V_k)$ is an (ε, d)-regular tuple as stated in the lemma. We choose t to be the largest odd number satisfying $t \leqslant \lfloor \varepsilon^2 km/(k-\ell) \rfloor$ and we want to cover $\mathcal{V}$ by ℓ-paths each having t edges. To this end, let S_{k-1} denote the symmetric group and for each permutation $\tau \in S_{k-1}$ let

$$\mathcal{V}(\tau) = (V_{\tau(1)}, V_{\tau(2)}, \ldots, V_{\tau(k-1)}, V_k).$$

Let p_0 denote the maximal integer for which there exists a family of pairwise disjoint ℓ-paths with exactly t edges each, such that every ℓ-path is canonical with respect to some $\mathcal{V}(\tau)$, $\tau \in S_{k-1}$, and for every $\tau \in S_{k-1}$ there are either exactly p_0 or $p_0 + 1$ paths in this family which are canonical with respect to $\mathcal{V}(\tau)$. Among those families let $\mathscr{P}_{p_0}$ be one with maximal cardinality and for each $\tau \in S_{k-1}$ for which there are $p_0 + 1$ canonical ℓ-paths with respect to $\mathcal{V}(\tau)$ in $\mathscr{P}_{p_0}$ we remove one of those paths to obtain $\mathscr{P} \subset \mathscr{P}_{p_0}$ with size $|\mathscr{P}| = p_0(k-1)!$. We will prove that $\mathscr{P}$ is the family of ℓ-paths required in the lemma.

For a family $\mathscr{P}'$ of paths let $V(\mathscr{P}') = \bigcup_{\mathcal{P} \in \mathscr{P}'} V(\mathcal{P})$ and we claim that there is an $\tilde{r} \in [k]$ such that $|V_{\tilde{r}} \setminus V(\mathscr{P}_{p_0})| < 2k\varepsilon m$. In the opposite case we pick $W_r \subset V_r \setminus V(\mathscr{P}_{p_0})$ with size $|W_r| = 2k\varepsilon m$ for all $r \in [k]$ and from regularity of $(V_1, \ldots, V_k)$ and $W_r \subset V_r$ we derive that

$$e(W_1, \ldots, W_k) \geqslant (d-\varepsilon)(2k\varepsilon m)^k.$$

Since $d \geqslant 2\varepsilon$ it follows from Proposition 19 that for any $\tau \in S_{k-1}$ there is a canonical ℓ-path with respect to $(W_{\tau(1)}, \ldots, W_{\tau(k-1)}, W_k)$ which consists of more than $\varepsilon^2 km/(k-\ell) \geqslant t$ edges. (Note that these ℓ-paths are not necessarily disjoint for different τ.) However, we get a contradiction either to the maximality of p_0 or to the maximality of $|\mathscr{P}_{p_0}|$.

Thus, with $U_r = V_r \cap V(\mathscr{P})$ for all $r \in [k]$, we derive that there exists an $\tilde{r} \in [k]$ such that

$$|U_{\tilde{r}}| \geqslant |V_{\tilde{r}}| - |\mathscr{P}_{p_0} \setminus \mathscr{P}|t - 2k\varepsilon m \geqslant |V_{\tilde{r}}| - 3k\varepsilon m,$$

since $|\mathscr{P}_{p_0} \setminus \mathscr{P}| \leqslant (k-1)!$, $t \leqslant \varepsilon^2 km/(k-\ell)$, and $\varepsilon \leqslant 1/k!$.

From the above we want to derive that

$$|U_r| \geqslant |V_r| - 7k\varepsilon m \quad \text{for all } r \in [k] \tag{16}$$

which would imply the lemma, since $\varepsilon \leqslant \beta/(7k^2)$.

To this end, note first that canonical ℓ-paths with t edges intersect sets in middle position in exactly t vertices, whereas sets in end positions are intersected in $(t+1)/2$ vertices (see Remark 18). Hence, for all $r \in [k-1]$ we have

$$\begin{aligned}|U_r| &= p_0\big[(k-2\ell)(k-2)!t + (2\ell-1)(k-2)!(t+1)/2\big] \\ &= p_0\big[(2k-2\ell-1)(k-2)!(t+1)/2 - (k-2\ell)(k-2)!\big]\end{aligned}$$

and

$$|U_k| = p_0(k-1)!(t+1)/2.$$

Suppose $\tilde{r} \neq k$ then $|U_r| = |U_{\tilde{r}}| \geqslant |V_{\tilde{r}}| - 3k\varepsilon m$ for all $r \in [k-1]$ and

$$p_0 \geqslant \frac{2}{(t+1)} \frac{|U_{\tilde{r}}|}{(2k-2\ell-1)(k-2)!}.$$

However, this implies

$$|U_k| \geqslant \frac{(k-1)|U_{\tilde{r}}|}{2k-2\ell-1} \geqslant (k-1)m - 3k\varepsilon m = |V_k| - 3k\varepsilon m.$$

On the other hand, if $\tilde{r} = k$ then

$$p_0 = \frac{2}{(t+1)} \frac{|U_k|}{(k-1)!}$$

from which we derive

$$|U_r| \geqslant (2k-2\ell-1)m - 7k\varepsilon m = |V_k| - 7k\varepsilon m$$

due to $m \geqslant m_0 \geqslant 2\varepsilon^{-3}$. In both cases, we obtain (16).

Lastly, note that $p_0(k-1)!(t+1)/2 \leqslant |V_k| = (k-1)m$ from which we infer $|\mathscr{P}| \leqslant 2k/\varepsilon^2 = p$. $\square$

4.3. *Proof of the Path-cover Lemma*

In this section we prove the Lemma 7.

Proof of Lemma 7. Given k, ℓ with $k > 2\ell$ and $\gamma, \varepsilon > 0$. We apply Lemma 20 with k, ℓ, $d = \gamma/6$ and $\beta = \varepsilon/3$ to obtain ε_{20}, p_{20} and m_{20} and subsequently apply Lemma 11 with k, ℓ, $\varepsilon_{11} = (\varepsilon/3)^{(k-1)}/5$ to obtain n_{11}. Finally, we apply Theorem 14 with k and

$$\varepsilon_{14} = \frac{1}{2}\min\left\{\frac{\gamma^2}{16}, \frac{\gamma}{24k}, \varepsilon_{11}^2, \frac{\varepsilon_{20}}{2k}\right\} \quad \text{and} \quad t_{14} = \max\left\{n_{11}, \frac{16k}{\varepsilon_{14}}\right\}$$

to obtain T_{14} and n_{14}. Let $p = T_{14}p_{20}$ and $n_0 \geqslant \max\{2k^2T_{14}/\varepsilon_{14}, n_{14}\}$ sufficiently large.

For a hypergraph $\mathcal{H}$ on $n \geqslant n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geqslant (\frac{1}{2(k-\ell)} + \gamma)n$ we apply the weak hypergraph regularity lemma (Theorem 14) with k, ε_{14} and t_{14}. By possibly moving at most $t(2k-2\ell-1)(k-1) < \varepsilon_{14}n$ vertices to V_0 we obtain an $2\varepsilon_{14}$-regular partition $V = V_0 \,\dot\cup\, V_1 \,\dot\cup\, V_2 \,\dot\cup \cdots \dot\cup\, V_t$ of $\mathcal{H}$ such that the partition classes satisfy

$$|V_1| = \cdots = |V_t| = (2k-2\ell-1)(k-1)m$$

for some positive integer m. Clearly, $|V_0| \leqslant 2\varepsilon_{14}n \leqslant \varepsilon n/3$ and $n/t \geqslant |V_i| \geqslant n/(2t)$ for all $i \in [t]$.

For the k-uniform cluster hypergraph $\mathcal{K} = \mathcal{K}(2\varepsilon_{14}, \gamma/6)$ of $\mathcal{H}$ on the vertex set $[t]$ we know by Proposition 16 that all but at most $\sqrt{2\varepsilon_{14}}t^{k-1} \leqslant \varepsilon_{11}t^{k-1}$ of the $(k-1)$-sets $S \in \binom{[t]}{k-1}$ satisfy

$$\deg_{\mathcal{K}}(S) \geqslant \left(\frac{1}{2(k-\ell)} + \frac{\gamma}{4}\right)t.$$

Thus, by Lemma 11 we find a $\mathcal{F}_{k,\ell}$-packing in $\mathcal{K}$ which covers all but at most $(5\varepsilon_{11})^{1/(k-1)}t \leqslant \varepsilon t/3$ vertices of $\mathcal{K}$.

Let $\mathcal{F}$ be an arbitrary copy of $\mathcal{F}_{k,\ell}$ in the cluster hypergraph $\mathcal{K}$ with the vertex set, say, $V(\mathcal{F}) = \{1, 2, \ldots, (2k-2\ell)(k-1)\}$ grouped into sets $A_1, \ldots, A_{2k-2\ell-1}, B$, all of the same size $k-1$. The edges of $\mathcal{F}$ are the sets $A_i \cup \{b\}$ with $i \in [2k-2\ell-1]$ and $b \in B$. We will show that the corresponding induced hypergraph $\mathcal{H}_{\mathcal{F}} = \mathcal{H}[V_1 \,\dot\cup\, V_2 \,\dot\cup \cdots \dot\cup\, V_{(2k-2\ell)(k-1)}]$ can be covered by a family of at most $(2k-2\ell-1)(k-1)p_{20}$ pairwise disjoint ℓ-paths which leave at most

$$(2k-2\ell-1)(k-1)\beta m \tag{17}$$

vertices of $\mathcal{H}_{\mathcal{F}}$ uncovered. This would imply that the union of these families for the $\mathcal{F}_{k,\ell}$-packing contains at most $tp_{20} \leqslant p$ pairwise disjoint ℓ-paths and the number of vertices in $\mathcal{H}$ not covered by these ℓ-paths is at most

$$|V_0| + (\varepsilon t/3) \times n/t + t\beta m \leqslant \varepsilon n,$$

as stated in the lemma.

To find a family of ℓ-paths satisfying (17) let $i \in [2k-2\ell-1]$ and by suppressing the dependence on i let $a_1, \ldots, a_{k-1}$ be the elements of A_i. For each $i \in [2k-2\ell-1]$ and each $a \in A_i$ we subdivide V_a into $(k-1)$ pairwise disjoint sets $U_a^1, \ldots, U_a^{k-1}$, each having

$$\frac{|V_a|}{k-1} = (2k-2\ell-1)m$$

vertices and, subsequently group them into tuples $(U_{a_1}^r, \ldots, U_{a_{k-1}}^r)$ with $r \in [k-1]$. Moreover, for all $b \in B$ we subdivide V_b into $(2k-2\ell-1)$ pairwise disjoint sets, each of size

$$\frac{|V_b|}{(2k-2\ell-1)} = (k-1)m.$$

Thus, we obtain $(2k-2\ell-1)(k-1)$ such sets and there is a bijection between those sets and the $(k-1)$-tuples $(U_{a_1}^r, \ldots, U_{a_{k-1}}^r)$. We fix such a bijection (arbitrarily) and denote the preimage of $(U_{a_1}^r, \ldots, U_{a_{k-1}}^r)$ by W_i^r (recall that we suppressed the dependence of $a_1, \ldots, a_{k-1}$ on i).

For each $i \in [2k-2\ell-1]$ and each $b \in B$ the set $A_i \cup \{b\}$ forms an edge in $\mathcal{K}$, i.e. the tuple $(V_{a_1}, \ldots, V_{a_{k-1}}, V_b)$ is $(2\varepsilon_{14}, \gamma/6)$-regular. Due to Fact 13 and $2\varepsilon_{14} \leqslant \varepsilon_{20}/2k$ we derive that the k-tuples $(U_{a_1}^r, \ldots, U_{a_{k-1}}^r, W_i^r)$ are all $(\varepsilon_{20}, \gamma/6)$-regular. Hence, for each $i \in [2k-2\ell-1]$ and each $r \in [k-1]$ we can apply Lemma 20 to $(U_{a_1}^r, \ldots, U_{a_{k-1}}^r, W_i^r)$ to obtain a family of at most p_{20} pairwise disjoint ℓ-paths which cover all but at most βm vertices of $(U_{a_1}^r, \ldots, U_{a_{k-1}}^r, W_i^r)$. Since there are exactly $(2k-2\ell-1)(k-1)$ such k-tuples we obtain at most $(2k-2\ell-1)(k-1)p_{20}$ paths in total and the number of vertices in $H_{\mathcal{F}}$ not covered by those paths is at most $(2k-2\ell-1)(k-1)\beta m$, as stated in (17). $\square$

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